Square Root Property, Completing the Square, Quadratic Formula
The Square Root Property: We know that if $x^2 = a$, $x = \sqrt{a}$ or $x = -\sqrt{a}$.

1. Use the square root property to solve the following.

$x^2 = 50$

$x = \sqrt{50}$ or $x = -\sqrt{50}$

$x = 5\sqrt{2}$ or $x = -5\sqrt{2}$

$16p^2 = 49$

$p^2 = \frac{49}{16}$

$p = \sqrt{\frac{49}{16}}$ or $p = -\sqrt{\frac{49}{16}}$

$p = 7/4$ or $p = -\frac{7}{4}$

$3u^2 + 4 = 31$

$3u^2 = 27 \rightarrow u^2 = 9$

$u = \sqrt{9}$ or $u = -\sqrt{9}$

$u = 3$ or $u = -3$

$5x^2 + 125 = 0$

$5x^2 = -125$

$x^2 = -25$

$x = -\sqrt{-25}$ or $x = \sqrt{-25}$

$x = -5i$ or $x = 5i$

$(p-5)^2 = 9$

$p - 5 = \sqrt{9}$ or $p - 5 = -\sqrt{9}$

$p = 5 + 3$ or $p = 5 - 3$

$p = 8$ or $p = 2$

$(3t-4)^2 = 4$

$2(x-3)^2 - 6 = 0$

$(x-3)^2 = 3$

$(x-3) = \sqrt{3}$ or $(x-3) = -\sqrt{3}$

$x = \sqrt{3} + 3$ or $x = 3 - \sqrt{3}$

$(u+5)^2 + 18 = 0$

$(u+5)^2 = -18$

$u + 5 = \sqrt{-18}$ or $u + 5 = -\sqrt{-18}$

$u = -5 + 3\sqrt{2}i$ or $u = -5 - 3\sqrt{2}i$

Notice the second row all had a binomial, like $(x + d)$ squared, which allowed us to use the square root property, since that was the only occurrence of the variable. If we have a quadratic equation that is not in this form, we can rewrite to be in this form, and this is called **completing the square**.

In fact, we can use completing the square to derive the **Quadratic Formula**, which shows that if

$$ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. Use completing the square, the Quadratic Formula, or factoring to solve the following.

$p^2 + 4p + 6 = 0$

$x_1, x_2 = \frac{-4 \pm \sqrt{16 - 4(1)(6)}}{2(1)} = \frac{-4 \pm \sqrt{8}}{2}$

$= -2 \pm \sqrt{2}$

$-3y - 10 = -y^2 \rightarrow y^2 - 3y - 10 = 0$

$x_1, x_2 = \frac{-(-3) \pm \sqrt{9 - 4(1)(-10)}}{2(1)}$

$= \frac{3 \pm \sqrt{49}}{2} = \frac{3 \pm 7}{2}$ $\nearrow x_1 = 5$ $\searrow x_2 = -2$

$2x(x+6) = 14 \rightarrow 2x^2 + 12x - 14 = 0$

$x_1, x_2 = \frac{-12 \pm \sqrt{144 - 4(2)(14)}}{2(2)}$

$= \frac{-12 \pm \sqrt{32}}{4} = -3 \pm \sqrt{2}$

$\frac{1}{5}h^2 + h + \frac{3}{5} = 0$

$x_1, x_2 = \frac{-1 \pm \sqrt{1 - 4(\frac{1}{5})(\frac{3}{5})}}{\frac{2}{5}} = \frac{-1 \pm \sqrt{13/25}}{\frac{2}{5}} = \frac{-1 \pm \frac{1}{5} \cdot \sqrt{13}}{2/5} = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}$

$x^2 + 4x + 8 = 0$

$x_1, x_2 = \frac{-4 \pm \sqrt{16 - 4(1)(8)}}{2(1)} = \frac{-4 \pm \sqrt{-16}}{2}$

$= \frac{-4 \pm 4i}{2} = -2 \pm 2i$

3. Remember that x-intercepts happen when $y = 0$ or $f(x) = 0$, and y-intercepts occur when $x = 0$. Find the x and y-intercepts of the functions below. Write your answer in point form.

$g(x) = 4x^2 + 8x - 5$ y-int: -5

x-int $g(x) = 0 \rightarrow 4y^2 + 8y - 5 = 0$

$x_1, x_2 = \frac{-8 \pm \sqrt{64 - 4(4)(-5)}}{2(4)} = \frac{-8 \pm 12}{8} = -\frac{2}{2} \pm \frac{3}{2}$

$= \frac{1}{2}, -\frac{5}{2}$

$f(x) = 3x^2 + 2x - 2$ y-int: -2

x-int: $f(x) = 0 \rightarrow 3x^2 - 2x - 2 = 0$

$x_1, x_2 = \frac{-(-2) \pm \sqrt{4 - 4(3)(-2)}}{2(3)} = \frac{2 \pm \sqrt{28}}{2 \cdot 3} = \frac{1}{3} \pm \frac{\sqrt{7}}{3} = \frac{1 \pm \sqrt{7}}{3}$

4. A baseball is thrown upward with an initial velocity of 32 $\frac{ft}{sec}$ from a cliff that is 48 feet off the ground. The baseball's height h (in feet) after t seconds is given by $h(t) = -16t^2 + 32t + 48$. Find the time at which the height of the ball is 64 feet.

$v_0 = 32$ ft/sec

$h = 48$ ft

$64 = -16t^2 + 32t + 48 \Rightarrow -16t^2 + 32t + 48 - 64 = 0$

$-16t^2 + 32t - 16 = 0$

$t = \frac{-(+32) \pm \sqrt{(32)^2 - 4(-16)(-16)}}{2(-16)} = \frac{-32}{-32} = 1$

at t=1s the height is 64 feet.

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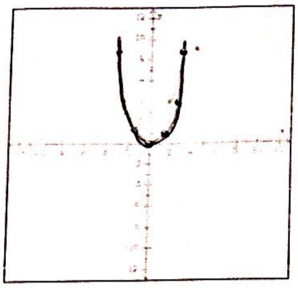
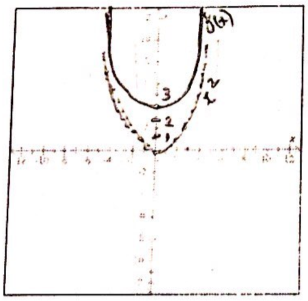
5. Sketch a graph of $y = x^2$ by plotting points, using $x = -3, -2, -1, 0, 1, 2, 3$.



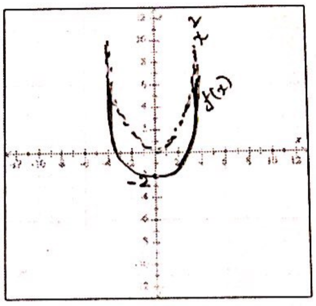
x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

6. Sketch the following graphs by plotting points, and compare these to the graph of $y = x^2$.

$f(x) = x^2 + 3$



$g(x) = x^2 - 2$

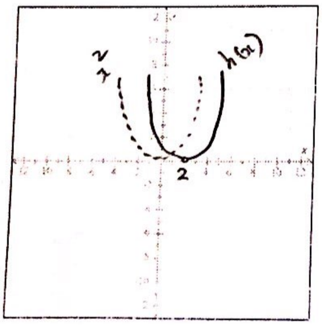


7. Generally, if $f(x) = x^2 + k$, describe how the graph shifts the $y = x^2$ if $k > 0$ and if $k < 0$.

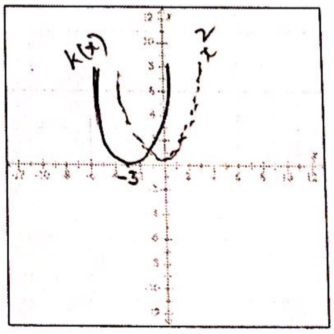
$f(x) = x^2 + k \begin{cases} \text{if } k>0, \text{ it shifts } k \text{ to up} \\ \text{if } k<0, \text{ it shifts } k \text{ to down} \end{cases}$

8. Sketch a graph of the following functions, and compare these to the graph of $y = x^2$.

$h(x) = (x - 2)^2$



$k(x) = (x + 3)^2$



9. Generally, if $f(x) = (x - h)^2$, describe how the graph shifts the $y = x^2$ if $h > 0$ and if $h < 0$.

$\begin{cases} \text{if } h>0, \text{ graph shifts } h \text{ to the right} \\ \text{if } h<0, \text{ graph shifts } h \text{ to the left} \end{cases}$

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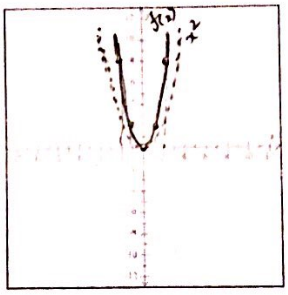
Graphs of Quadratic Equations

1. Sketch the following graphs by plotting points, and compare these to the graph of $y = x^2$.

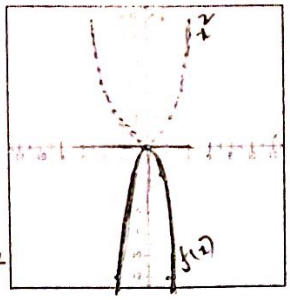
$f(x) = 2x^2$

x	y
-3	18
-2	8
-1	2
0	0
1	2
2	8
3	18



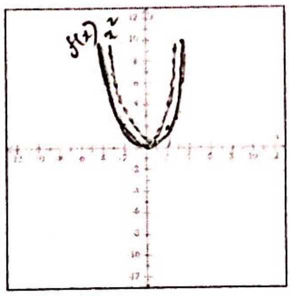
$g(x) = -3x^2$

x	y
-3	-27
-2	-12
-1	-3
0	0
1	-3
2	-12
3	-27



$h(x) = \frac{1}{2}x^2$

x	y
-3	9/2
-2	4/2
-1	1/2
0	0
1	1/2
2	4/2
3	9/2



2. Generally, if $f(x) = ax^2$, describe how a affects the graph of $y = x^2$ if $0 < a < 1$, if $a > 1$, if $-1 < a < 0$, or if $a \leq -1$.

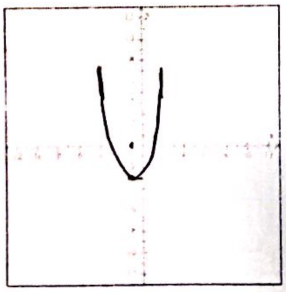
if $0<a<1$ Parabola opens upward and streches,
if $a>0$ Parabola opens upward and Compresses,
if $-1<a<0$ Parabola opens downward and streches.
if $a \leq -1$ Parabola opens downward and Compresses.

3. Putting this all together, we see that a quadratic equation of the form $f(x) = a(x-h)^2 + k$, which is called the **Vertex Form**:

- Is the graph of $y = x^2$ shifted h spaces to the right or left (depending on the sign of h), and shifted k spaces up or down (depending on the sign of k). This means that its **vertex** is (h, k).
- The axis of symmetry is $x = h$.
- If $a > 0$, the parabola opens upward, and k is the minimum value of the function.
- If $a < 0$, the parabola opens downward, and k is the maximum value of the function.

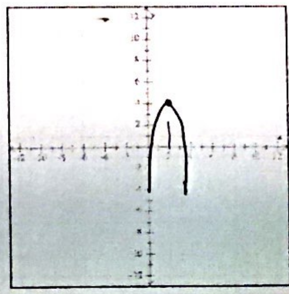
4. Graph the following. First state their vertex and axis of symmetry. Also, state whether the function has a minimum or a maximum and what that value is.

$f(x) = (x+1)^2 - 3$



Vertex: $(-1, -3)$
axis of Symmetry: $x = -1$
$k = -3$ is the minimum

$g(x) = -2(x-2)^2 + 4$



Vertex: $(+2, 4)$
axis of Symmetry: $x = 2$
$k = 4$ is the maximum

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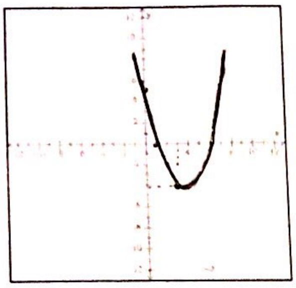
Quadratic equations will not always be written in the vertex form. We can use completing the square to see that when a quadratic equation is in standard form, $ax^2 + bx + c$, the vertex is given by $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

5. For the following quadratic function, find the vertex, x and y-intercepts, and axis of symmetry. State whether it opens up or down and has a minimum or maximum. Where (at what x value) does the min/max occur, and what is that min/max value. Sketch a graph using what you have found.

$f(x) = x^2 - 6x + 5$



vertex : $-\frac{(-6)}{2} = 3 \rightarrow (3,-4)$

$f(3) = 9 - 18 + 5 = -4$

x-int : $x^2 - 6x + 5 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(5)}}{2} = \frac{6 \pm 4}{2} = 5, 1$ $(5,0), (1,0)$

y-int : 5 $(0,5)$

6. Ben sells used iPhones. The average cost to package iPhones is given by the equation $C(x) = 3x^2 - 120x + 1300$, where x is the number of iPhones packaged per month.

Determine the number of iPhones that Ben needs to package in a month to minimize the average cost.

$C'(x) = 0 \Rightarrow 3(2)x - 120 + 0 = 0$

$\rightarrow x = \frac{120}{6} = \boxed{20}$

or $\frac{-b}{2a} = \frac{120}{2(3)} = \boxed{20}$

What is the minimum cost?

$C(20) = 3(20)^2 - 120(20) + 1300 = \boxed{100}$

7. An arrow is shot straight upward into the air from the ground with an initial velocity of 128 $\frac{ft}{sec}$. The height of the arrow off the ground (in feet) is represented by $h(t) = -16t^2 + 128t$, where t is the number of seconds after it is shot. Answer the following.

- At what time does the arrow reach its max height?

$h'(t) = 0 \Rightarrow -32t + 128 = 0$

$t = \frac{128}{32} = 4s$

or $x = \frac{-128}{-16 \times 2} = \boxed{4s}$

- What is its max height?

$h(4) = -16(4)^2 + 128(4) = \boxed{256 ft}$

- When will it reach the ground again?

$h(t) = -16t^2 + 128t = 0$

$t(-16t + 128) = 0$

$t = 0$

$t = \frac{128}{16} = \boxed{8s}$

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Composition of Functions and Functions and Inverses

$f(x) = x + 4$ $\qquad g(x) = 2x^2 + 4x$ $\qquad h(x) = x^2 + 1$ $\qquad k(x) = \dfrac{1}{x}$

1. Given the functions above, find the following.

$(f + g)(x)$

$= f(x) + g(x)$

$= x + 4 + 2x^2 + 4x = \boxed{2x^2 + 5x + 4}$

$(g - f)(x)$

$g(x) - f(x)$

$= 2x^2 + 4x - 4 - x$

$= \boxed{2x^2 + 3x - 4}$

$(g \cdot k)(x)$

$[g(x)] \cdot [k(x)]$

$= [2x^2 + 4x] \cdot \frac{1}{x}$

$= \boxed{2x + 4}$

$\left(\dfrac{h}{k}\right)(x)$

$\dfrac{h(x)}{k(x)} = \dfrac{x^2+1}{\frac{1}{x}} = \boxed{x^3 + x}$

$(h \circ f)(x) = h(f(x))$

$(x+4)^2 + 1 = x^2 + 8x + 16 + 1$

$= \boxed{x^2 + 8x + 17}$

$(k \circ h)(x) = k(h(x))$

$= \boxed{\dfrac{1}{x^2+1}}$

$(f \circ g)(x) = f(g(x))$

$= \boxed{2x^2 + 4x + 4}$

We defined a function by saying that it is a relation that assigns each x (domain) value exactly one y (range) value. We are now going to look at **Inverse Functions**. The Inverse Function, $f^{-1}(x)$ of a function $f(x)$ exchanges the domain and range of $f(x)$, meaning that the domain of $f(x)$ becomes the range of f^{-1}, and the range of $f(x)$ becomes the domain for $f^{-1}(x)$.

2. If the following points for a function $f(x)$ represents the pounds of coffee sold as x and the total for that coffee as y, $f = [(1, 8.50), (4, 34), (1.5, 12.75)]$. This means that 1 pound of coffee costs \$8.50, 4 pounds cost \$34, and 1.5 pounds costs \$12.75.

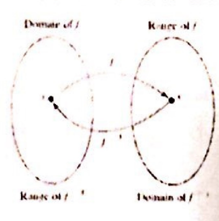
Find f^{-1} and state the meaning of the points.

$f^{-1} = [(8.50, 1), (34, 4), (12.75, 1.5)]$

The first point means that 8.5 pound of coffee cost \$1. And so on...!

Since we interchange x and y values for the inverse function of a function, then for a function to have an inverse, if the x values in two coordinate pairs are different, then the y values must also be different. Remember we had the vertical line test to test if y is a function of x, now we will use the **Horizontal Line Test** to see if f is **one-to-one**, which is necessary for it to have an inverse.

Now if f maps x to y and f^{-1} maps y back to x, see picture below, then $f(f^{-1}(x)) = x = f^{-1}(f(x))$. So inverse functions undo each other and get us back to x, very handy!



3. Verify that f and g are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$. g(x) is not given!

$f(x) = 6x + 1$ $\qquad f(x) = \dfrac{\sqrt[3]{x}}{2}$

Since inverse functions exchange x and y values, to find an inverse function of a one-to-one function, $y = f(x)$, we can follow these steps:

- Replace $f(x)$ with y.
- Interchange x and y.
- Solve for y.
- Replace y with $f^{-1}(x)$.

4. Use the steps above to find an equation for the inverse of each of the one-to-one functions below.

$f(x) = \frac{1}{3}x - 2$

$y = \frac{1}{3}x - 2$

$x = \frac{1}{3}y - 2$

$x + 2 = \frac{1}{3}y \rightarrow 3(x+2) = y \rightarrow \boxed{f^{-1}(x) = 3x + 6}$

$n(x) = 4x + 2$

$y = 4x + 2$

$x = 4y + 2 \rightarrow \frac{x-2}{4} = y$

$\boxed{f^{-1}(x) = \frac{x-2}{4}}$

$h(x) = \dfrac{4x - 1}{3}$

$y = \frac{4x-1}{3}$

$x = \frac{4y-1}{3} \rightarrow 3x = 4y - 1$

$\frac{3x+1}{4} = y \rightarrow \boxed{f^{-1}(x) = \frac{3x+1}{4}}$

$g(x) = x^3 + 1$

$y = x^3 + 1 \rightarrow x = y^3 + 1$

$\rightarrow x - 1 = y^3 \rightarrow y = \sqrt[3]{x-1}$

$\boxed{f^{-1}(x) = \sqrt[3]{x-1}}$

$k(x) = 4\sqrt[3]{x - 5}$

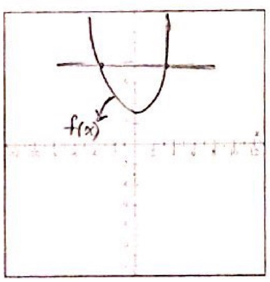
$y = 4\sqrt[3]{x-5}$

$x = 4\sqrt[3]{y-5} \rightarrow (\frac{x}{5})^3 = (\sqrt[3]{y-5})^3$

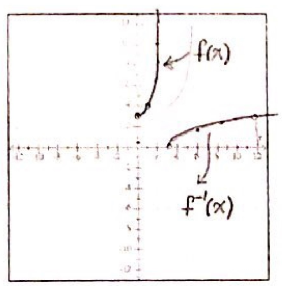
$\frac{x^3}{125} = y - 5 \rightarrow y = \frac{x^3}{125} + 5 \rightarrow \boxed{f^{-1}(x) = \frac{x^3}{125} + 5}$

We use the horizontal line test to see if a function is one-to-one. If a function fails this test, we may be able to restrict its domain so that it is one-to-one.

5. Sketch a graph of $f(x) = x^2 + 3$. Does it pass the horizontal line test? If not, how can we restrict its domain (which x-values can we limit or function to) so that it is one-to-one? Sketch the new graph (restricted domain) on the second coordinate plane.



Not one-to-one



if domain: $[0, +\infty)$

then f(x) is one-to-one

So, $f(x) = x^2 + 3$, $x \geq 0$

$y = x^2 + 3$

$x = y^2 + 3 \rightarrow (x-3) = y^2 \rightarrow \sqrt{x-3} = y$

$f^{-1}(x) = \sqrt{x-3}$

Label some of your points on your restricted domain, and since the inverse function will interchange x and y values of this function, plot points for the inverse function on the same coordinate plane and sketch its graph.

6. Now look back at problem 5. State the domain and range of both f and f^{-1}. Find an equation for $f^{-1}(x)$.

Domain $f(x)$: $x \geq 0$

Range $f(x)$: $y \geq 3$

Domain $f^{-1}(x)$: $x \geq 3$

Range $f^{-1}(x)$: $x \geq 0$

$\boxed{f^{-1}(x) = \sqrt{x-3}}$

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